

# REMARK ON MEROMORPHIC FUNCTIONS THAT SHARE FIVE PAIRS

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ABSTRACT. We determine all pairs  $(f, g)$  of meromorphic functions that share four pairs of values  $(a_\nu, b_\nu)$ ,  $1 \leq \nu \leq 4$ , and a fifth pair  $(a_5, b_5)$  under some mild additional condition.

## 1. INTRODUCTION

Meromorphic functions  $f$  and  $g$  are said to share the pair  $(a, b)$  of complex numbers (including  $\infty$ ), if  $f - a$  and  $g - b$  ( $1/f$  and  $1/g$ , if  $a = \infty$  and  $b = \infty$ , respectively) have the same zeros. Czubiak and Gundersen [1] proved that meromorphic functions  $f$  and  $g$  that share *six* pairs  $(a_\nu, b_\nu)$  are Möbius transformations of each other, hence share all pairs  $(a, L(a))$  for some Möbius transformation  $L$ . On the other hand, the functions

$$(1) \quad \hat{f}(z) = \frac{e^z + 1}{(e^z - 1)^2} \quad \text{and} \quad \hat{g}(z) = \frac{(e^z + 1)^2}{8(e^2 - 1)}$$

share the values  $\infty, 0, 1$ , and  $-\frac{1}{8}$  with different multiplicities, and the pair  $(-\frac{1}{2}, \frac{1}{4})$  counting multiplicities. Thus

$$(2) \quad f(z) = \frac{1}{\hat{f}(z) + \frac{1}{2}} \quad \text{and} \quad g(z) = \frac{1}{\hat{g}(z) - \frac{1}{4}}$$

are not Möbius transformations of each other and share the pairs  $(0, 0)$ ,  $(2, -4)$ ,  $(\frac{2}{3}, \frac{4}{3})$  and  $(\frac{8}{3}, -\frac{8}{3})$  with different multiplicities, and the value  $\infty$  (the pair  $(\infty, \infty)$ ) counting multiplicities. Moreover,  $f$  and  $g$  have common counting function of poles  $\overline{N}(r, \infty) = T(r) + S(r)$ , where  $T(r)$  and  $S(r)$  denote the common Nevanlinna characteristic and remainder term of  $f$  and  $g$  (for notations and results of Nevanlinna theory the reader is referred to Hayman's monograph [5]), and  $f$  and  $g$  parametrise the algebraic curve

$$(3) \quad 4x^2 + 2xy + y^2 - 8x = 0.$$

Gundersen's example  $\hat{f}, \hat{g}$  was the first to show that in Nevanlinna's *Four Value Theorem* [7] one cannot dispense with the condition 'counting multiplicities' for each of the four values. This is possible for one value (Gundersen [2]) and also for two of the values (Gundersen [3], Mues [6]), while the case of three such values is still open. The state of art is outlined in [10]. Gundersen's example also has another characterisation due to Reinders [8, 9]: *If  $f$  and  $g$  share four values  $a_\nu$ ,*

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2000 *Mathematics Subject Classification.* 30D35.

*Key words and phrases.* Nevanlinna theory, value- and pair-sharing, four-value-theorem, five-pairs-theorem.

and if  $f^{-1}(a) \subset g^{-1}(b)$  holds for some pair  $(a, b)$  ( $a, b \neq a_\nu$ ), then either  $f$  and  $g$  are Möbius transformations of each other or else  $f = T \circ \hat{f} \circ h$  and  $g = T \circ \hat{g} \circ h$  holds for some Möbius transformation  $T$  and some non-constant entire function  $h$ .

In [4], Gundersen considered functions  $f$  and  $g$  that share five pairs and are not Möbius transformations of each other. He proved several sharp inequalities for the corresponding Nevanlinna functions, including  $T(r, f) = T(r, g) + S(r)$  and

$$(4) \quad \overline{N}(r; a_\nu, b_\nu) \geq \frac{1}{3}T(r) + S(r),$$

where  $\overline{N}(r; a_\nu, b_\nu)$  denotes the counting functions of common  $(a_\nu, b_\nu)$ -points of  $(f, g)$ , not counting multiplicities, and  $T(r)$  and  $S(r)$  denote the common Nevanlinna characteristic and remainder term, respectively.

## 2. MAIN RESULT

The aim of this paper is to prove

**THEOREM 1.** *Suppose that meromorphic functions  $f$  and  $g$  share four pairs  $(a_\nu, b_\nu)$ , and a fifth pair  $(a_5, b_5)$  counting multiplicities and such that*

$$(5) \quad m(r, 1/(f - a_5)) + m(r, 1/(g - b_5)) = S(r)$$

*holds. Then either  $f$  and  $g$  are Möbius transformations of each other or else  $f = T \circ \hat{f} \circ h$  and  $g = S \circ \hat{g} \circ h$  holds for suitably chosen Möbius transformations  $S$  and  $T$  and some non-constant entire function  $h$ .*

*Proof.* We note that (5) is automatically fulfilled if  $a_\nu = b_\nu$ ,  $1 \leq \nu \leq 4$ . Three of the pairs  $(a_\nu, b_\nu)$  may be prescribed. We will assume  $(a_1, b_1) = (0, 0)$ ,  $(a_2, b_2) = (2, -4)$ , and, in particular,  $a_5 = b_5 = \infty$ , to stay as close as possible with the modified example of Gundersen. Then  $f$  and  $g$  have the same poles counting multiplicities, and such that

$$(6) \quad m(r, f) + m(r, g) = S(r)$$

holds. We also assume that  $f$  and  $g$  are not Möbius transformations of each other. Similar to the approach in [4] we consider

$$P(x, y, \mathbf{c}) = c_1 x^2 + c_2 xy + c_3 y^3 + c_4 x + c_5 y.$$

Then there are at least two linear independent vectors  $\mathbf{c} = (c_1, \dots, c_5) \in \mathbb{C}^5$  such that

$$(7) \quad P(a_\nu, b_\nu, \mathbf{c}) = 0 \quad (1 \leq \nu \leq 4)$$

holds, that is,  $P(z) = P(f(z), g(z), \mathbf{c})$  vanishes whenever  $f(z) = a_\nu$  and  $g(z) = b_\nu$ . If  $P$  does not vanish identically, this yields

$$\sum_{\nu=1}^4 \overline{N}(r; a_\nu, b_\nu) \leq \overline{N}(r, 1/P) \leq T(r, P) + O(1) \leq 2T(r) + S(r);$$

for the last inequality the additional hypothesis (6) is used. On the other hand it follows from the Second Main Theorem that

$$\sum_{\nu=1}^4 \overline{N}(r; a_\nu, b_\nu) + \overline{N}(r, \infty) \geq 3T(r) + S(r),$$

hence  $T(r) \leq \overline{N}(r, \infty) + S(r)$ . Thus, still assuming  $P \neq 0$ , it follows that

$$\begin{aligned} \overline{N}(r, 1/P) &= N(r, 1/P) + S(r) = 2T(r) + S(r) \\ m(r, 1/P) &= S(r) \\ \sum_{\nu=1}^4 \overline{N}(r, a_\nu, b_\nu) &= \overline{N}(r, 1/P) + S(r) \\ T(r) &= \overline{N}(r, \infty) + S(r). \end{aligned}$$

In particular, the quotient  $\chi(z) = P(z; \tilde{\mathbf{c}})/P(z; \mathbf{c})$  satisfies  $T(r, \chi) = S(r)$ . In other words,  $f$  and  $g$  parametrise the algebraic curve

$$(8) \quad F(x, y; z) = \chi_1 x^2 + \chi_2 yx + \chi_3 y^2 + \chi_4 x + \chi_5 y = 0 \quad (\chi_k = \chi c_k - \tilde{c}_k)$$

over the field  $\mathbb{C}(\chi)$ . This is also true if  $P(z; \mathbf{c})$  or  $P(z; \tilde{\mathbf{c}})$  vanishes identically. It is obvious that  $\chi_1 \chi_3 \neq 0$ , since otherwise  $g$  [resp.  $f$ ] would be a Möbius transformation or a rational function of  $f$  [resp.  $g$ ] of degree two over the field  $\mathbb{C}(\chi)$ . In the first case it would follow that  $g$  is an ordinary Möbius transformation of  $f$ , while in the second case we would obtain a contradiction:  $T(r, g) = 2T(r, f) + S(r)$ .

The algebraic curve (8) has the rational parametrisation (set  $x = ty$ )

$$x = \frac{p(z, t)}{s(z, t)} = -\frac{t(\chi_4 t + \chi_5)}{\chi_1 t^2 + \chi_2 t + \chi_3}, \quad y = \frac{q(z, t)}{s(z, t)} = -\frac{\chi_4 t + \chi_5}{\chi_1 t^2 + \chi_2 t + \chi_3}$$

with  $t = x/y$ . In terms of  $f$  and  $g$  this yields

$$\begin{aligned} f(z) &= \frac{p(z, \mathbf{t}(z))}{s(z, \mathbf{t}(z))} = -\frac{\mathbf{t}(z)(\chi_4 \mathbf{t}(z) + \chi_5)}{\chi_1 \mathbf{t}(z)^2 + \chi_2 \mathbf{t}(z) + \chi_3} \\ g(z) &= \frac{q(z, \mathbf{t}(z))}{s(z, \mathbf{t}(z))} = -\frac{\chi_4 \mathbf{t}(z) + \chi_5}{\chi_1 \mathbf{t}(z)^2 + \chi_2 \mathbf{t}(z) + \chi_3} \end{aligned} \quad \text{with} \quad \mathbf{t}(z) = \frac{f(z)}{g(z)}.$$

Since by (4),  $f$  and  $g$  have ‘many’ zeros, there are three possibilities to be discussed: The zeros correspond to the

- a) *poles of  $\mathbf{t}$* , in which case  $\chi_4 \equiv 0$  and ‘almost all’ zeros of  $f$  are simple, while the zeros of  $g$  have order two. Moreover,  $\mathbf{t}$  has ‘almost no’ zeros ( $N(r, 1/\mathbf{t}) = S(r)$ ).
- b) *zeros of  $\mathbf{t}$* , in which case  $\chi_5 \equiv 0$  and ‘almost all’ zeros of  $g$  are simple, while the zeros of  $f$  have order two. Moreover,  $\mathbf{t}$  has ‘almost no’ poles ( $N(r, \mathbf{t}) = S(r)$ ).
- c) *zeros of  $\chi_4(z)\mathbf{t}(z) + \chi_5(z)$  with  $\chi_4 \chi_5 \neq 0$* . Then ‘almost all’ zeros of  $f$  and  $g$  are simple, and  $\mathbf{t}$  has ‘almost no’ zeros and poles ( $N(r, 1/\mathbf{t}) + N(r, \mathbf{t}) = S(r)$ ).

Taking all pairs  $(a_\nu, b_\nu)$  ( $1 \leq \nu \leq 4$ ), into account, the following holds: for every  $\nu$  there exist  $\phi_\nu, \psi_\nu, \alpha_\nu, \beta_\nu, \tilde{\beta}_\nu \in \mathbb{C}(\chi)$  such that  $p(z, t) - a_\nu s(z, t) = \phi_\nu(t - \alpha_\nu)(t - \beta_\nu)$  and  $q(z, t) - b_\nu s(z, t) = \psi_\nu(t - \alpha_\nu)(t - \tilde{\beta}_\nu)$ , respectively; occasionally the factor  $(t - \beta_\nu)$  and  $(t - \tilde{\beta}_\nu)$  corresponding to  $\beta_\nu \equiv \infty$  and  $\tilde{\beta}_\nu \equiv \infty$ , respectively, might be missing. The functions<sup>1</sup>  $\alpha_\nu$  are mutually distinct, and the same is true for  $\beta_\nu$  and also  $\tilde{\beta}_\nu$ . It is also obvious that  $\beta_\nu \neq \tilde{\beta}_\nu$ , and that both functions are exceptional for  $\mathbf{t}$ , except when one of them coincides with  $\alpha_\nu$ . Since  $\mathbf{t}$  has at most two exceptional functions, we obtain the following picture:

<sup>1</sup>At first glance one would expect that  $\alpha_\nu, \beta_\nu, \tilde{\beta}_\nu$  are algebraic over  $\mathbb{C}(\chi)$ . But this is not the case, since analytic continuation which permutes  $\alpha_\nu$  and  $\beta_\nu$  would also permute  $\alpha_\nu$  and  $\tilde{\beta}_\nu$ , in contrast to  $\beta_\nu \neq \tilde{\beta}_\nu$ .

For  $\nu = 1$  and  $\nu = 4$ , say, we have  $\beta_\nu \equiv \alpha_\nu$ , that is, the pairs  $(a_\nu, b_\nu)$ , are attained by  $(f, g)$  in a  $(2 : 1)$  manner, while for  $\nu = 2$  and  $\nu = 3$  this happens the other way  $(1 : 2)$ . This means that, in addition to (8), that also

$$(9) \quad F_y(a_\nu, b_\nu; z) \equiv 0 \quad (\nu = 1, 4) \quad \text{and} \quad F_x(a_\nu, b_\nu; z) \equiv 0 \quad (\nu = 2, 3)$$

holds. To stay close with the modified example of Gundersen we assume  $\chi_3 \equiv 1$  (this is possible since  $\chi_3 \not\equiv 0$  is already known). From (9), that is

$$\chi_5 = 4\chi_1 - 4\chi_2 + \chi_4 = 2\chi_1 a_3 + \chi_3 b_3 + \chi_4 = \chi_2 a_4 + 2b_4 \equiv 0,$$

one can compute the coefficients  $\chi_k$  in terms of  $a_3, b_3, a_4, b_4$ , namely

$$(10) \quad \chi_1 = \frac{b_4(b_3 + 4)}{a_4(a_3 - 2)}, \quad \chi_2 = -\frac{2b_4}{a_4}, \quad \chi_3 = 1, \quad \chi_4 = \frac{2b_4(2b_3 + 4a_3)}{a_4(2 - a_3)}.$$

In particular, the functions  $\chi_k$  are constant, and  $f$  and  $g$  are rational functions (now over  $\mathbb{C}$ ) of the meromorphic function  $t = f/g$ . Having determined the coefficients (10) we now use (8) to express  $b_3$  and  $b_4$  in terms of  $a_3$  and  $a_4$ . The solutions to  $F(a_\nu, b_\nu; z) = 0$  for  $\nu = 2, 4$  are given by<sup>(2)</sup>

- $b_4 = -2a_4, b_3 = -2a_3$ , while  $F(a_3, b_3; z) = 0$  is automatically fulfilled; this leads to  $g = -2f$ .

- $b_4 = 2a_4 - 8$  and  $b_3 = \frac{2(8 - 4a_4 + a_4 a_3)}{a_4 - 4}$ . Since, however,  $F(a_3, b_3; z) = 32 \frac{(a_3 - 2)(a_4 - 2)(a_3 - a_4 + 2)}{(a_4 - 4)^2}$  also has to vanish, we just have to discuss the sub-case  $a_3 = a_4 - 2$ , since  $a_3 = 2$  and also  $a_4 = 2$  would contradict  $a_2 = 2$ . Thus  $a_3 = a_4 - 2, b_3 = 2a_4 - 4$  and  $b_4 = 2a_4 - 8$ , and  $(a_1, a_2, a_3, a_4)$  and  $(b_1, b_2, b_3, b_4)$  have the same cross-ratio  $\frac{a_1 - a_3}{a_2 - a_3} : \frac{a_1 - a_4}{a_2 - a_4} = \frac{b_1 - b_3}{b_2 - b_3} : \frac{b_1 - b_4}{b_2 - b_4} = \frac{(a_4 - 2)^2}{a_4(a_4 - 4)}$ . In other words, there exists some Möbius transformation  $L$  such that  $f$  and  $L \circ g$  share four values  $a_1, a_2, a_3, a_4$  and the pair  $(\infty, L(\infty))$ . By Reinders' characterisation this implies  $f = T \circ \hat{f} \circ h$  and  $g = S \circ \hat{g} \circ h$ , where  $S$  and  $T$  are suitably chosen Möbius transformations, and  $h$  is some non-constant entire function.  $\square$

FINAL REMARK. It remains open whether or not—and how—the hypothesis (6) may be relaxed. Is it sufficient to assume that the pair  $(a_5, b_5)$  is shared ‘counting multiplicities’ by  $f$  and  $g$ ? Is it even true that functions sharing five pairs are either Möbius transformations of each other or else have the form  $f = T \circ \hat{f} \circ h$  and  $g = S \circ \hat{g} \circ h$ ?

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<sup>2</sup>We note that `maple` was not able to determine all solutions to the system (7), but note also that the coefficients  $\chi_k$  are functions of the  $a_\nu, b_\nu$ ; to get an impression: one has to solve

$$(b_3 + 4)b_4 + (8 - 4a_3)a_4 = a_3^2 b_4(b_3 + 4) + b_3^2(2 - a_3) = a_4(b_3 + 4) + a_3(8 - b_4) + 2b_4 + 4b_3 = 0$$

for  $b_3, b_4$ .

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